

Heights and totally real numbers

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Abstract

1973 Schinzel proved in [Sch73] that the standard logarithmic height h on the maximal totally real field extension of the rationals is either zero or bounded from below by a positive constant. In this paper we study this property for canonical heights associated to rational functions and the corresponding dynamical system on the affine line. At the end, we will give a few remarks on the behavior of h on finite extensions of the maximal totally real field.

1 Introduction

We fix the following notations. Let h be the standard logarithmic height on the algebraic numbers. We fix an algebraic closure $\overline{\mathbb{Q}}$ and we denote the maximal totally real algebraic subfield by \mathbb{Q}^{tr} . Further we say that a field $F \subset \overline{\mathbb{Q}}$ has the Bogomolov property relative to a height function h' if and only if $h'(\alpha)$ is either zero or bounded from below by a positive constant for all $\alpha \in F$. Now we can state the theorem of Schinzel as follows.

Theorem 1.1 (Schinzel). *The field \mathbb{Q}^{tr} has the Bogomolov property relative to h .*

This was the first example of a field having infinite degree over \mathbb{Q} with the Bogomolov property. The proof of Schinzel gives the sharp lower bound $\frac{1}{2} \log(\frac{1+\sqrt{5}}{2})$. If one is interested only in a quantitative result, one can use Bilu's equidistribution theorem (see [BG], Theorem 4.3.1) as follows.

Proof: Assume there is a sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ in $\mathbb{Q}^{tr} \setminus \{-1, 0, 1\}$, such that the height of these points tends to zero. Then, by Bilu's equidistribution theorem, the equidistributed probability measures on the set of conjugates of the α_i converge weakly to the probability measure on the unit circle. But the support of each such Galois measure lies in the real line. Hence they cannot cover the unit circle, which leads to a contradiction. \square

See also [HS93] for a very short proof of Schinzel's original result. Roughly speaking, the field \mathbb{Q}^{tr} seems to be arithmetically "easy". In this paper we will give a complete classification of rational functions defined over the algebraic numbers such that \mathbb{Q}^{tr} has the Bogomolov property relative to the canonical height coming from this rational map. Our result reads as follows.

Theorem 1.2. *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree at least two. Then the following statements are equivalent:*

- i) \mathbb{Q}^{tr} has the Bogomolov property relative to \widehat{h}_f .*
- ii) There is a $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that the Julia set of $\sigma(f)$ is not contained in \mathbb{R} .*
- iii) The set $\text{PrePer}(f) \cap \mathbb{Q}^{tr}$ is finite.*

In section 2 we give a short introduction to canonical dynamical heights and state a few facts on Julia sets that will be needed in the proof of Theorem 1.2. Section 3 contains a proof of a partial result of our main theorem. This result is completely covered by Theorem 1.2, but the proof is very simple and shows the strategy for proving Theorem 1.2 very clearly. In the same section, we also give a necessary condition for a rational map to have a real Julia set. In section 4 we proof the main theorem and give an additional equivalence in the case of a polynomial. One class of polynomials with real Julia set are Chebyshev polynomials. We will study these polynomials in section 5 and give a nice Bogomolov property relative to the canonical height coming from these polynomials. The last section is independent from the rest of this paper. There we give a short proof of a theorem of Amoroso and Nuccio concerning lower bounds for h in CM-fields and answer a question that was stated in their paper. Their result implies that there is no positive lower bound for the height h on $\mathbb{Q}^{tr}(i)$. Therefore, we summarize in this section briefly the known results on the behavior of h in finite extensions of \mathbb{Q}^{tr} .

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2 Heights and dynamics

Canonical heights associated to rational functions defined over the algebraic numbers can be defined using the next theorem due to Call and Silverman.

Theorem 2.1. *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree greater one. There is a unique height function \widehat{h}_f , called the canonical height related to f , such that for all $\alpha \in \overline{\mathbb{Q}}$ we have*

$$i) \widehat{h}_f(f(\alpha)) = \deg(f) \widehat{h}_f(\alpha) \quad \text{and} \quad ii) \widehat{h}_f = h + O(1).$$

See [Si07], Chapter 3.4, for a proof and additional information. In fact, we work with a rational function on the Riemann sphere which we identify with $\mathbb{C} \cup \{\infty\}$. On the Riemann sphere, we will always use the complex topology which is induced by the chordal metric ρ . The Julia set of such a map f is the set of points where f acts "chaotically".

Definition. Let f be a self map of the Riemann sphere. The Fatou set $F(f)$ of f is the maximal open subset of the Riemann sphere, satisfying the condition: For all $\alpha \in F(f)$ and all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\rho(\alpha, \beta) < \delta \Rightarrow \rho(f^n(\alpha), f^n(\beta)) < \varepsilon$$

for all $n \in \mathbb{N}$. The Julia set $J(f)$ of f is the complement of $F(f)$.

In addition to the canonical height associated with a rational function f of degree ≥ 2 there exists a f -invariant canonical probability measure μ_f which is supported on the Julia set of f (see [FLM83]). Replacing Bilu's theorem by a dynamical equidistribution theorem (see [Yu08], Theorem 3.7) we can generalize Theorem 1.1 as follows.

Theorem 2.2. Let $f \in \overline{\mathbb{Q}}(x)$ be a rational function of degree > 1 such that the Julia set $J(f)$ of f is not contained in the real line. Then \mathbb{Q}^{tr} has the Bogomolov property relative to \widehat{h}_f . Furthermore there are only finitely many preperiodic points in \mathbb{Q}^{tr} .

Notice that $h = \widehat{h}_{x^2}$ and the Julia set of the map x^2 is indeed the unit circle. Hence Schinzel's result (in an ineffective version) is a special case of Theorem 2.2.

Remark. Theorem 2.2 also includes a special case of a theorem of Zhang ([Zh98], Corollary 2). Let E be an elliptic curve defined over a number field K with Néron-Tate height \widehat{h}_E . Then there exists a rational map $f \in K(x)$, called Lattès map, such that $\widehat{h}_E(P) = \frac{1}{2}\widehat{h}_f(x(P))$ for all $P \in E(\overline{\mathbb{Q}})$. Further, the Julia set of f is the complete Riemann sphere. Hence Theorem 2.2 tells us that there is a positive constant c such that $\widehat{h}_E(P) \geq c$ for all non-torsion points $P \in E(\mathbb{Q}^{tr})$ and there are only finitely many torsion points in $E(\mathbb{Q}^{tr})$. Notice that there is an effective constant c in the case where K is totally real (see [BP05], Theorem 17).

We will collect some important facts on Julia sets of rational maps.

Facts 2.3. Let $f \in \mathbb{C}(x)$ be a rational function of degree at least two. Then we have

- a) $J(f)$ is not empty,
- b) $J(f)$ is completely invariant, i.e. $f(J(f)) = f^{-1}(J(f)) = J(f)$,
- c) there are no isolated points in $J(f)$,
- d) $J(f)$ is the closure of the repelling periodic points of f .

Proof: For proofs of these statements we refer to [Be], Theorem 4.2.1, Theorem 3.2.4, Theorem 5.7.1 and Theorem 6.9.2. \square

3 A first example

A natural question arising from Theorem 2.2 is the following. Does \mathbb{Q}^{tr} have the Bogomolov property relative to \hat{h}_f even if $J(f)$ is contained in the real line? Before proving our main theorem which gives a complete answer to this question, we will give a counterexample. One class of rational maps with real Julia set is given by the quadratic polynomials $f_c(x) = x^2 - c$, where $c \geq 2$ is a real number.

Proposition 3.1. *Let c be a rational with $c \geq 2$. Then \mathbb{Q}^{tr} does not have the Bogomolov property relative to \hat{h}_{f_c} .*

Proof: Take an $\epsilon \in (-c, c) \cap \mathbb{Q}$, such that ϵ is no preperiodic point of f_c . This is possible by Northcott's theorem. We will prove that for each $n \in \mathbb{N}$ the set $f^{-n}(\epsilon)$ is contained in \mathbb{Q}^{tr} . Then for all n we take an arbitrary γ_n in $f^{-n}(\epsilon)$ and get a sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ in \mathbb{Q}^{tr} , with $\hat{h}_{f_c}(\gamma_n) = \frac{1}{2^n} \hat{h}_{f_c}(\epsilon)$. This tends to zero, proving the claim.

Since c is in \mathbb{Q} we have $\sigma(f^n(\gamma)) = f^n(\sigma(\gamma))$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and all algebraic numbers γ . Thus we know that the set $f^{-n}(\epsilon)$ is galois invariant. Hence it suffices to show that $f^{-n}(\epsilon)$ is contained in the real line in order to prove the proposition.

We will prove this by induction over n . For $n = 1$ we have $f^{-1}(\epsilon) = \pm\sqrt{\epsilon + c}$, which is real by the choice of ϵ . Furthermore $|\pm\sqrt{\epsilon + c}| < c$, since $c \geq 2$ and $|\epsilon| < c$. Now assume that for a given $n \in \mathbb{N}$ we have $f^{-n}(\epsilon) \subset \mathbb{R}$ and $|\gamma| < c$ for all $\gamma \in f^{-n}(\epsilon)$. Every $\beta \in f^{-(n+1)}(\epsilon)$ is in $f^{-1}(\gamma)$ for a $\gamma \in f^{-n}(\epsilon)$. So, it is of the form $\beta = \pm\sqrt{\gamma + c}$. We conclude exactly as in the case $n = 1$ that we have $\beta \in \mathbb{R}$ and $|\beta| < c$. \square

Remark. Indeed, the above argumentation proves that $J(f_c)$ is real for $c \geq 2$. Notice therefor that such a f_c has a repelling fixed point, and thus an element of $J(f_c)$, in the interval $(-c, c)$. We conclude as above that the backward orbit of this repelling fixed point lies in \mathbb{R} . As the backward orbit of every element in $J(f_c)$ is dense in $J(f_c)$ (see [Be], Theorem 4.2.7 ii)), we get $J(f_c) \subseteq \mathbb{R}$. If c is in $\overline{\mathbb{Q}} \cap \mathbb{R}$, $c \geq 2$, the same proof shows that $\mathbb{Q}(c)^{tr}$ has not the Bogomolov property relative to \hat{h}_{f_c} .

The proof of the main theorem is just like the above one but uses some abstract results on Julia sets that lie in a circle on the Riemann sphere.

Every σ in $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ extends to a unique endomorphism of $\overline{\mathbb{Q}}(x)$ with $\sigma(x) = x$. So we can define the rational map $\sigma(f)$ for all $f \in \overline{\mathbb{Q}}(x)$ and all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Lemma 3.2. *Let $f \in \overline{\mathbb{Q}}(x)$ be a rational map of degree > 1 . Then we have $\hat{h}_f = \hat{h}_{\sigma(f)} \circ \sigma$.*

Proof: This follows directly from the definition and the trivial facts $\deg(f) = \deg(\sigma(f))$ and $\sigma(f(\alpha)) = \sigma(f)(\sigma(\alpha))$ for all $\alpha \in \overline{\mathbb{Q}}$. \square

Lemma 3.3. *Let $f \in \mathbb{C}(x)$ be a rational map of degree ≥ 2 . If the Julia set $J(f)$ of f is contained in the real line, then f has a representative with only real coefficients.*

Proof: Choose the representative of f with monic enumerator. Then we can write $f(x) = \frac{A(x)+iB(x)}{C(x)+iD(x)}$ with real valued polynomials A, B, C and D such that $A(x)+iB(x)$ and $C(x)+iD(x)$ have no common zeros. As $A(x)+iB(x)$ was assumed to be monic, we know that A is not equal to zero and $\deg(B) < \deg(A)$. For $r \in \mathbb{R}$ we have $f(r)$ is real if and only if $A(r)D(r) - B(r)C(r) = 0$. Applying the Facts 2.3 to our assumption $J(f) \subset \mathbb{R}$ we know that f takes infinitely many real elements to real elements. Hence we have the equation $A(x)D(x) = B(x)C(x)$. We want to show that this can only occur if $D(x) = B(x) = 0$, this means only if $f \in \mathbb{R}(x)$.

Assume that $B(x) \neq 0$. Then also $D(x) \neq 0$ and we have $0 \leq \deg(B) < \deg(A)$. Thus $A(x)$ cannot be a divisor of $B(x)$ and so a greatest common divisor $R_1(x)$ of $A(x)$ and $C(x)$ is not constant. Write $A(x) = R_1(x)R_2(x)$ and $C(x) = R_1(x)R_3(x)$. By the maximality of $R_1(x)$ we see that $R_2(x)$ is a divisor of $B(x)$ and $R_3(x)$ is a divisor of $D(x)$. The equation $A(x)D(x) = B(x)C(x)$ gives us a polynomial $R_4(x)$, such that $B(x) = R_2(x)R_4(x)$ and $D(x) = R_3(x)R_4(x)$. This leads to the equations

$$A(x) + iB(x) = R_1(x)R_2(x) + iR_2(x)R_4(x) = R_2(x)(R_1(x) + iR_4(x))$$

$$C(x) + iD(x) = R_1(x)R_3(x) + iR_3(x)R_4(x) = R_3(x)(R_1(x) + iR_4(x)).$$

$R_1(x)$ is not constant and so both polynomials have a common zero, what was excluded by assumption. Hence $B(x) = D(x) = 0$ and $f \in \mathbb{R}(x)$. \square

4 Proof of the main result

Now we are prepared to proof our main theorem.

Theorem 4.1. *As usual let $f \in \overline{\mathbb{Q}}(x)$ be a rational map of degree at least two. Then the following statements are equivalent:*

- i) \mathbb{Q}^{tr} has the Bogomolov property relative to \hat{h}_f .
- ii) There is a $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, such that the Julia set $J(\sigma(f))$ is not contained in \mathbb{R} .
- iii) The set $\text{PrePer}(f) \cap \mathbb{Q}^{tr}$ is finite.

Proof: Notice again that $J(f)$ cannot be empty, see Fact 2.3 a). By Theorem 2.2, we conclude easily that ii) yields i) and iii). Assume there is a $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, such that $J(\sigma(f))$ is not contained in the real line. If i) or ii) are wrong, then there is a sequence of pairwise distinct elements $\{\alpha_n\}_{n \in \mathbb{N}}$ in \mathbb{Q}^{tr} with $\hat{h}_f(\alpha_n) \rightarrow 0$ for $n \rightarrow \infty$. Then, by Lemma 3.2, $\{\sigma(\alpha_n)\}_{n \in \mathbb{N}}$ is an infinite sequence in \mathbb{Q}^{tr} with canonical height $\hat{h}_{\sigma(f)}(\sigma(\alpha_n))$ tends to zero. This is not possible due to Theorem 2.2.

The implication iii) \Rightarrow ii) is not hard either. Assume $J(\sigma(f))$ is contained in the real line for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. By Fact 2.3 d) $J(f)$ contains all repelling periodic points of f . For all maps $\sigma(f)$, $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there are only finitely many non-repelling periodic points (see [Be], §9.6). Hence there are infinitely many points

$\alpha \in \overline{\mathbb{Q}}$ such that $\sigma(\alpha)$ is a repelling periodic point of $\sigma(f)$, for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It follows from our assumption that all these α are totally real. Especially we get $|\text{PrePer}(f) \cap \mathbb{Q}^{tr}| = \infty$.

Finally, we prove that *i*) implies *ii*). Assume again that $J(\sigma(f))$ is contained in the real line for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then Lemma 3.3 tells us that $f \in K(x)$ for a totally real number field K . Let $\sigma_1, \dots, \sigma_d$ be a complete set of embeddings of K into $\overline{\mathbb{Q}}$. For each $\sigma_i(f)$ there exists a finite set of intervals, such that all backward orbits of these intervals again lie in this finite set of intervals. This nice result can be found in [EvS11], Theorem 2 and the discussion afterwards. Thus for all σ_i we can choose a real interval (a_i, b_i) , such that for all $c \in (a_i, b_i)$ every backward orbit is contained in the real line. For all σ_i take a $c_i \in (a_i, b_i) \cap \mathbb{Q}$ and choose a global $\varepsilon > 0$ such that $(c_i - \varepsilon, c_i + \varepsilon) \subset (a_i, b_i)$ for all $1 \leq i \leq d$. All the σ_i give rise to non equivalent absolute values on K . By the approximation theorem of Artin and Whaples (see [La], Chapter I, §1), there exists a $c \in K$, such that $|\sigma_i(c - c_i)| = |\sigma_i(c) - c_i| < \varepsilon$. This implies that $\sigma_i(c)$ lies in the interval (a_i, b_i) for all σ_i . Notice therefor that c is totally real. There are infinitely many points c with this property in K , but as a number field K contains only finitely many preperiodic points of f . Thus we can assume that c is no preperiodic point of f .

For every γ with $f^n(\gamma) = c$ we have $\sigma(f)^n(\sigma(\gamma)) = \sigma(c)$, $n \in \mathbb{N}$. From the choice of our intervals it follows that all conjugates of γ are in the real line, and hence we can conclude $f^{-n}(c) \subset \mathbb{Q}^{tr}$. Now choose for all $n \in \mathbb{N}$ a γ_n in $f^{-n}(c)$. This gives a sequence $\{\gamma_n\}$ in \mathbb{Q}^{tr} , such that

$$0 \neq \widehat{h}_f(\gamma_n) = \frac{1}{\deg(f)^n} \widehat{h}_f(c) \rightarrow 0 \quad .$$

Notice that we have chosen a non preperiodic c . This shows that \mathbb{Q}^{tr} cannot have the Bogomolov property relative to \widehat{h}_f . \square

In the case where $f \in \overline{\mathbb{Q}}[x]$ is a polynomial we can give a further nice equivalence.

Theorem 4.2. *Let $f \in \overline{\mathbb{Q}}[x]$ be a polynomial. Then the following statements are equivalent:*

- i) \mathbb{Q}^{tr} has not the Bogomolov property relative to \widehat{h}_f*
- ii) $J(\sigma(f)) \subset \mathbb{R}$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*
- iii) $\text{PrePer}(f) \subset \mathbb{Q}^{tr}$*
- iv) $\widehat{h}_f(\alpha) > 0$ for all $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}^{tr}$*

Proof: *i*) and *ii*) are equivalent by Theorem 4.1 and the equivalence of *iii*) and *iv*) is trivial. As we have a polynomial, the Julia set of f is the boundary of the set

$$\{y \in \mathbb{C} \mid |f^n(y)| \rightarrow \infty, \text{ as } n \rightarrow \infty\} \quad .$$

See [Mi], Lemma 9.4. This set is called the filled Julia set of f . Every preperiodic point of f is contained in the filled Julia set of f . For a polynomial it follows from the

definitions of the Julia set and the filled Julia set that ∞ is in neither of both sets. Hence, both sets are bounded. Now assume *ii*). Then the Julia set of every $\sigma(f)$ is a closed subset of a closed interval I . The only bounded subset of the Riemann sphere with such a boundary is the set itself. This means that all these $J(\sigma(f))$ coincides with its filled Julia set and hence it contains $\text{PrePer}(\sigma(f)) = \sigma(\text{PrePer}(f))$. This shows that $\text{PrePer}(f)$ is contained in \mathbb{Q}^{tr} .

Now we assume *iii*). Then $\text{PrePer}(\sigma(f)) = \sigma(\text{PrePer}(f))$ lies in the real line for every $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence, the closure of the repelling periodic points of $\sigma(f)$ lies in the real line. By Fact 2.3 *d*) the closure of the repelling periodic points is the Julia set. This concludes the proof. \square

5 Chebyshev polynomials and open questions

Let's go back to the quadratic polynomial $f_c = x^2 - c \in \overline{\mathbb{Q}}[x]$. We have seen in Proposition 3.1 that the canonical height h_{f_c} can get arbitrary small on \mathbb{Q}^{tr} for every rational $c \geq 2$. This behavior may change completely for non rational c . If c is smaller than $-\frac{1}{4}$, then it has a non real fixed point. This implies, by Theorem 4.2, that \mathbb{Q}^{tr} has the Bogomolov property relative to \hat{h}_{f_c} when $c = \sqrt{q} \geq 2$, where q is in $\mathbb{Q} \setminus \mathbb{Q}^2$. Notice that in this case $\sigma(x^2 - \sqrt{q}) = x^2 + \sqrt{q}$ for a $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and the non real fixed point of $\sigma(x^2 - \sqrt{q})$ is a conjugate of a fixed point of $x^2 - \sqrt{q}$. In the set of rational maps over \mathbb{Q} with real Julia set is one special class, namely the Chebyshev polynomials. For example the polynomial f_2 from Proposition 3.1 is the second Chebyshev polynomial. This means that the following diagram commutes.

$$\begin{array}{ccc} \overline{\mathbb{Q}}^* & \xrightarrow{x \mapsto x^d} & E \\ \varphi \downarrow & & \downarrow \varphi \\ \overline{\mathbb{Q}} & \xrightarrow{T_d} & \overline{\mathbb{Q}} \end{array} \quad (1)$$

Here φ is given by the map $x \mapsto x + x^{-1}$. Thus the Chebyshev-polynomials have an underlying algebraic structure. So they are in some sense similar to Lattès maps, where the underlying structure is given by an elliptic curve. An interesting fact we can deduce from Theorem 4.1 is that \mathbb{Q}^{tr} has the Bogomolov property relative to the height h coming from the map $x \mapsto x^d$, but not relative to \hat{h}_{T_d} , although these heights are related in a very strong way. This relation can be easily made explicit.

Proposition 5.1. *For all $z \in \overline{\mathbb{Q}}^*$ we have $\hat{h}_{T_d}(z + z^{-1}) = 2h(z)$.*

Proof: As in (1) we define $\varphi(x) = x + x^{-1}$. We have to check that $\frac{1}{2}\hat{h}_{T_d} \circ \varphi$ fulfills the two conditions given in Theorem 2.1 for the canonical height $\hat{h}_{x^d} = h$. Using the commutativity of (1) we get $\frac{1}{2}\hat{h}_{T_d}(\varphi(z^d)) = \frac{1}{2}\hat{h}_{T_d}(T_d(\varphi(z))) = d\frac{1}{2}\hat{h}_{T_d}(\varphi(z))$. As φ has degree two, we also have $\frac{1}{2}\hat{h}_{T_d} \circ \varphi = \frac{1}{2}h \circ \varphi + O(1) = h + O(1)$. \square

Definition. *A Salem number is a real algebraic integer $\alpha > 1$ such that all conjugates of α have absolute value ≤ 1 and at least one conjugate has absolute value equal to 1.*

As one conjugate of the Salem number α has absolute value 1, the inverse of a conjugate is again a conjugate of α . This implies, using the definition, that α^{-1} is the only real conjugate of α and all other conjugates lie on the unit circle. Hence $\alpha + \alpha^{-1}$ is a totally real number.

We have seen that the Bogomolov property for \mathbb{Q}^{tr} does not hold relative to \widehat{h}_{T_d} , $d \geq 2$. The next best bound one can ask for is a bound of Lehmer strength. This means, one can ask whether there exists a positive constant $c > 0$ such that $\deg(\alpha)\widehat{h}_{T_d}(\alpha) \geq c$ for all α in $\mathbb{Q}^{tr} \setminus \text{PrePer}(T_d)$. This would be a quite strong result, because it would imply that the absolute value of a Salem number is bounded away from one. This follows from Proposition 5.1 and the fact that $\alpha + \alpha^{-1}$ is totally real for all Salem numbers α . On the other hand the existence of such a bound c seems to be very likely, as \mathbb{Q}^{tr} has the Bogomolov property relative to all $\widehat{h}_{f_2-\epsilon}$ for every algebraic $\epsilon > 0$.

Let K be a number field and denote with K^{ab} the maximal abelian field extension of K . Amoroso and Zannier proved in [AZ00] that K^{ab} has the Bogomolov property relative to h . Their result also implies the Bogomolov property of these fields relative to \widehat{h}_{T_d} .

Proposition 5.2. *Let $T_d(x)$ be the d -th Chebyshev polynomial, where d is at least 2. Let K be any number field. Then the field K^{ab} has the Bogomolov property relative to \widehat{h}_{T_d} .*

Proof: Let α be an arbitrary element in $K^{ab} \setminus \text{PrePer}(T_d)$. Take a pre-image β of α under the map $z \mapsto z + z^{-1}$. Then we have $[K^{ab}(\beta) : K^{ab}] \leq 2$. From the choice of α we know $h(\beta) \neq 0$, hence by Proposition 5.1 and [AZ00], Theorem 1.1, we get

$$\widehat{h}_{T_d}(\alpha) = 2h(\beta) \geq c(K) \left(\frac{\log 4}{\log \log 10} \right)^{-13},$$

for a constant $c(K) > 0$ only depending on the ground field K . \square

The result of Amoroso and Zannier we have used here is an extension of a theorem due to Amoroso and Dvornicich. Amoroso and Dvornicich proved in [AD99] that the maximal abelian field extension over \mathbb{Q} has the Bogomolov property relative to h (see [AD99]). This result can also be stated dynamically.

The field $\mathbb{Q}(\text{PrePer}(f))$ has the Bogomolov property relative to \widehat{h}_f , where $f = x^2$.

Kronecker's theorem and Proposition 5.1 show that the preperiodic points of T_d are given by the set $\{\zeta + \zeta^{-1} \mid \zeta \text{ root of unity}\}$. Hence $\mathbb{Q}(\text{PrePer}(T_d))$ is an abelian extension of \mathbb{Q} and Proposition 5.2 shows that the statement above also holds if f is a Chebyshev polynomial. An interesting question is now for which other rational maps f the above property is true. According to the results of Habegger in [Ha11] it seems to be very likely that it holds for Lattès maps f defined over the rational numbers.

6 Finite extensions of \mathbb{Q}^{tr}

We will study the surprising behavior of h in finite extensions of \mathbb{Q}^{tr} . In 2007 Amoroso and Nuccio proved that there are elements with arbitrary small height in the union of CM-fields. Remember that a CM-field is a quadratic totally imaginary field extension of a totally real number field. As in [ADZ11] we will give a very short direct proof of this theorem and answer a question of Amoroso and Nuccio, whether there exists such a sequence in the set of generators of normal CM-fields. We will negatively answer this latter question by proving the following theorem. The proof of this theorem is independent of the results from the previous sections of this paper.

Theorem 6.1. *There are algebraic numbers $\{\alpha_n\}_{n \in \mathbb{N}}$ such that $\mathbb{Q}(\alpha_n)$ is a normal CM-field, none of these elements is a root of unity and $h(\alpha_n)$ tends to zero.*

Like in [AN07] we will use the following characterization of CM-fields.

Lemma 6.2. *A number field K is a CM-field if and only if there exists an element α with $K = \mathbb{Q}(\alpha)$ and $|\sigma(\alpha)| = 1$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.*

See [AN07], Proposition 2.3, or [BL78], Theorem 1.

Proof of Theorem 6.1: Take an algebraic number α that is no root of unity and such that $|\sigma(\alpha)| = 1$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Assume furthermore that the degree of α over \mathbb{Q} is two (for example, set $\alpha = \frac{\sqrt{-15+1}}{4}$). Fix for all $n \in \mathbb{N}_0$ a 2^n -th root of α and denote it by $\alpha^{1/2^n}$. For all σ we have $1 = |\sigma(\alpha)| = |\sigma(\alpha^{1/2^n})|^{2^n}$. Hence $\mathbb{Q}(\alpha^{1/2^n})$ is a CM-field and we have $0 \neq h(\alpha^{1/2^n}) = \frac{1}{2^n} h(\alpha) \rightarrow 0$ for $n \rightarrow \infty$.

In what follows we will show that the extensions $\mathbb{Q}(\alpha^{1/2^n})/\mathbb{Q}$ are normal. We will prove this by induction over n . The case $n = 0$ is trivial as α was chosen to have degree two over \mathbb{Q} . Assume now that the field $\mathbb{Q}(\alpha^{1/2^n})$ is normal. To ease notation we set $\alpha' := \alpha^{1/2^n}$. Then we must show that $\mathbb{Q}(\sqrt{\alpha'})$ is normal, for a fixed square root of α' . The conjugates of $\sqrt{\alpha'}$ are given by the set $\{\pm\sqrt{\alpha'_i} \mid 1 \leq i \leq 2^n\}$, where α'_i runs over all conjugates of α' . As $\mathbb{Q}(\alpha')$ is normal we have $\alpha'_i \in \mathbb{Q}(\alpha') \subset \mathbb{Q}(\sqrt{\alpha'})$ for all $1 \leq i \leq 2^n$. Let β_i be a conjugate of $\sqrt{\alpha'}$ with $\beta_i^2 = \alpha'_i$. As $|\beta_i| = 1$, β_i cannot be real. Hence $\mathbb{Q}(\beta_i)$ is a quadratic extension of the maximal totally real subfield $\mathbb{Q}(\alpha')^+$ of $\mathbb{Q}(\alpha')$. This implies that the real minimal polynomial $x^2 - (\beta_i + \overline{\beta_i})x + 1$ of β_i must be defined over $\mathbb{Q}(\alpha')^+$. Thus $\beta_i + \overline{\beta_i}$ is in $\mathbb{Q}(\alpha')^+ \subset \mathbb{Q}(\sqrt{\alpha'})$. Multiplying $\beta_i + \overline{\beta_i}$ with $\alpha'_i = \beta_i^2$ leads to $\beta_i^3 + \beta_i = \beta_i(\alpha'_i + 1) \in \mathbb{Q}(\sqrt{\alpha'})$. α'_i is not equal to -1 and so β_i is an element of $\mathbb{Q}(\sqrt{\alpha'})$. This proves that $\mathbb{Q}(\sqrt{\alpha'})$ is normal. \square

The next immediate corollary is a quite simple example of the fact that the Bogomolov property is not preserved under finite field extensions. See also [ADZ11] for the same result.

Corollary 6.3. *$\mathbb{Q}^{tr}(i)$ does not have the Bogomolov property relative to h .*

Proof: This is due to the fact that $\mathbb{Q}^{tr}(i)$ is the compositum of all CM-fields. To see this we have to show that every CM-field is contained in $\mathbb{Q}^{tr}(i)$. Let therefore $\mathbb{Q}(\alpha)$

be a CM-field. By Lemma 6.2 we can assume that $|\sigma(\alpha)| = 1$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence $\alpha + \alpha^{-1}$ and $i(\alpha - \alpha^{-1})$ are totally real. Now we see $\alpha = \frac{1}{2}(\alpha + \alpha^{-1} + \alpha - \alpha^{-1}) \in \mathbb{Q}^{tr}(i)$. \square

Schinzel's original theorem from [Sch73] is much stronger than the formulation of Theorem 1.1. For example, he proved that the height of every non zero element in $\mathbb{Q}^{tr}(i)$ that does not lie on the unit circle is at least $\frac{1}{2} \log \left(\frac{\sqrt{5}+1}{2} \right)$. Moreover, [Sch73], Cor. 1, states the following.

Theorem 6.4 (Schinzel). *Let $\alpha \neq 0$ be an algebraic number with $D = [\mathbb{Q}^{tr}(\alpha) : \mathbb{Q}^{tr}]$ and such that the minimal polynomial P of α over \mathbb{Q}^{tr} is not reciprocal; i.e. $P(\alpha^{-1}) \neq 0$. Then we have*

$$h(\alpha) > \frac{1}{2D} \log \left(\frac{\sqrt{17}+1}{4} \right) .$$

Especially, this inequality is true for all elements in an extension field of \mathbb{Q}^{tr} of finite and odd degree D .

Proof: Let K be the field generated by the coefficients of P and then use [Sch73], Corollary 1, and Jensens formula. Notice that an irreducible reciprocal polynomial different from $x \pm 1$, must have even degree. \square

A natural question arising from Corollary 6.3 is: Does a finite extension K of \mathbb{Q}^{tr} has the Bogomolov property relative to h if $i \notin K$? This question remains unanswered in general, but John Garza has given an argument for a positive answer if K is not totally imaginary.

Proposition 6.5. *Let $K = \mathbb{Q}^{tr}(\alpha)$ such that α has at least one real conjugate. Denote the number of real conjugates of α with r_α and set $R_\alpha = \frac{r_\alpha}{\deg(\alpha)}$. Then we have*

$$h(\beta) \geq \frac{R_\alpha}{2} \log \left(\frac{2^{1-1/R_\alpha} + \sqrt{4^{1-1/R_\alpha} + 4}}{2} \right) ,$$

for all $\beta \in K^ \setminus \{\pm 1\}$.*

Proof: Let β be an arbitrary element in $K^* \setminus \{\pm 1\}$, and let L be a totally real number field with $\beta \in L(\alpha)$. We have $[L(\alpha) : \mathbb{Q}] = \deg(\alpha)[L(\alpha) : \mathbb{Q}(\alpha)]$. Furthermore, an embedding of $L(\alpha)$ is real if and only if it is an extension of a real embedding of $\mathbb{Q}(\alpha)$. Hence, the number of real embeddings of $L(\alpha)$ is $r_\alpha[L(\alpha) : \mathbb{Q}(\alpha)]$. Since β is in $L(\alpha)$, we also have $[L(\alpha) : \mathbb{Q}] = \deg(\beta)[L(\alpha) : \mathbb{Q}(\beta)]$, and a real embedding of $L(\alpha)$ must be an extension of a real embedding of $\mathbb{Q}(\beta)$. This means that the number of real embeddings of $L(\alpha)$ is at most $r_\beta[L(\alpha) : \mathbb{Q}(\beta)]$, with r_β defined as above. Together this yields

$$R_\beta = \frac{r_\beta[L(\alpha) : \mathbb{Q}(\beta)]}{[L(\alpha) : \mathbb{Q}]} \geq \frac{r_\alpha[L(\alpha) : \mathbb{Q}(\alpha)]}{[L(\alpha) : \mathbb{Q}]} = R_\alpha .$$

Now we can use [Ga07], Theorem 1, to achieve

$$h(\beta) \geq \frac{R_\alpha}{2} \log \left(\frac{2^{1-1/R_\alpha} + \sqrt{4^{1-1/R_\alpha} + 4}}{2} \right) .$$

A similar argumentation shows that R_α is independent from the choice of generator of K . \square

Remark. There is also a proof of Corollary 6.3 using dynamical methods and Theorem 4.1. The Möbius transformation $g(x) = \frac{x+i}{x-i}$ maps the real line onto the unit circle. Take the map $g^{-1} \circ x^2 \circ g$. By [Be], Theorem 3.1.4, we have $J(g^{-1} \circ x^2 \circ g) = g^{-1}(J(x^2)) = \mathbb{R}$. The same is true for the only galois conjugate $\frac{x-i}{x+i}$ of g . Furthermore, it is easy to check that we have $\widehat{h}_{g^{-1} \circ x^2 \circ g} = h \circ g$. Now Theorem 4.1 tells us that there are pairwise distinct totally real algebraic numbers $\{\alpha_j\}_{j \in \mathbb{N}}$ such that

$$0 \neq \widehat{h}_{g^{-1} \circ x^2 \circ g}(\alpha_j) = h(g(\alpha_j)) \rightarrow 0 .$$

As $g(\alpha_j)$ is in $\mathbb{Q}^{tr}(i)$ for all $j \in \mathbb{N}$, we conclude the corollary.

Notice that the field $\mathbb{Q}^{tr}(i)$ has the Bogomolov property relative to \widehat{h}_f , where f is a Lattès map defined over a totally real field ([Po12], Cor. 5).

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